ON THE STABILITY OF STEADY ROTATION OF A RIGID BODY WITH A FIXED POINT

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Steady rotations of a heavy rigid body with one fixed point, were first discussed by Mlodzeevskii [1] and Staude [2]. Sufficient conditions for their stability were given by Rumiantsev [3]. Beletskii [4] considered some cases of steady rotations and their stability in the Newtonian field. Apykhtin [5] found permanent axes of rotation when the equations of motion permit Goriachev's integrals.

In this paper, steady rotations of a rigid body with one fixed point and their stability in a potential field of force, are investigated.

1. Permanent axes of rotation. Let us assume a fixed point O belonging to the rigid body to be the origin of both, fixed coordinate system $O\xi\eta\zeta$ and a moving coordinate system $Ox_1x_2x_3$, the latter being fixed relative to the rigid body and its axes coinciding with the principal axes of inertia of this body with respect to the origin O. Let A_1, A_2, A_3 be the principal moments of inertia of the body with respect to the origin, and γ_1, γ_2, γ_3 — direction cosines of the axis $O\zeta$ with respect to x_1, x_2, x_3. We shall assume that the external forces permit the force function of the type

$$U = U(\gamma_1, \gamma_2, \gamma_3) \tag{1.1}$$

which has continuous partial derivatives of the 1-st order. Equations of motion in the moving reference frames are

$$A_i \frac{dp_i}{dt} = (A_{i+1} - A_{i+2})\, p_{i+1}p_{i+2} + \gamma_{i+2}\frac{\partial U}{\partial \gamma_{i+1}} - \gamma_{i+1}\frac{\partial U}{\partial \gamma_{i+2}} \tag{1.2}$$

$$\frac{d\gamma_i}{dt} = p_{i+2}\gamma_{i+1} - p_{i+1}\gamma_{i+2} \qquad (i = 1, 2, 3)$$

where p_i are the projections of the instantaneous angular velocity on the moving axes. The subscripts here should not exceed 3; this can be accomplished by taking the subscript modulus 3, i.e. substracting 3 if they exceed 3.

The system of differential equations (1.2) has the following first integrals:

$$A_1p_1^2 + A_2p_2^2 + A_3p_3^2 - 2U(\gamma_1, \gamma_2, \gamma_3) = \text{const}, \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{1.3}$$

$$A_1p_1\gamma_1 + A_2p_2\gamma_2 + A_3p_3\gamma_3 = \text{const},$$

If an axis of steady rotation exists, it will be stationary with respect to both, the body and a space around it. Let the direction cosines of this axis with reference to the moving frame be l_1 = const , l_2 = const , l_3 = const and ω be a corresponding angular velocity. Then we have $p_i = \omega l_i$, and (1.2) become

$$A_i l_i \frac{d\omega}{dt} = (A_{i+1} - A_{i+2})\,\omega^2 l_{i+1} l_{i+2} + \gamma_{i+2} \frac{\partial U}{\partial \gamma_{i+1}} - \gamma_{i+1} \frac{\partial U}{\partial \gamma_{i+2}}$$

$$\frac{d\gamma_i}{dt} = \omega\,(l_{i+2}\gamma_{i+1} - l_{i+1}\gamma_{i+2}) \qquad (1.4)$$

First integrals of this system are

$$\omega^2 (A_1 l_1^2 + A_2 l_2^2 + A_3 l_3^2) - 2U(\gamma_1, \gamma_2, \gamma_3) = k_1$$

$$\omega\,(A_1 l_1 \gamma_1 + A_2 l_2 \gamma_2 + A_3 l_3 \gamma_3) = k_2 \qquad (1.5)$$

$$l_1\gamma_1 + l_2\gamma_2 + l_3\gamma_3 = k_3, \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \qquad (1.6)$$

Eliminating ω from (1.5), we obtain

$$(A_1 l_1 \gamma_1 + A_2 l_2 \gamma_2 + A_3 l_3 \gamma_3)^2 [k_1 + 2U(\gamma_1, \gamma_2, \gamma_3)] = k_2^2 (A_1 l_1^2 + A_2 l_2^2 + A_3 l_3^2) \qquad (1.7)$$

Equations (1.6) and (1.7) can be used to determine γ_i. It is easily seen that this method will, in most cases, give constant values. Exceptions will occur when, e.g. some of the equations are found to have been derived from other equations not considered here. If γ_i is constant, then (1.4) gives

$$\frac{\gamma_1}{l_1} = \frac{\gamma_2}{l_2} = \frac{\gamma_3}{l_3}$$

Hence, $O\zeta$ will be a permanent axis, and $\gamma_i = l_i$. From (1.5) it follows that ω is also constant. It remains to determine the position of the permanent axis within the body, i.e. to find the conditions which must be fulfilled by l_i in order for the corresponding axis directed along $O\zeta$, to be the axis of steady rotation. If

$$\left(\frac{\partial U}{\partial \gamma_i}\right)_{\gamma_1 = l_1,\ \gamma_2 = l_2,\ \gamma_3 = l_3} = L_i \qquad (i = 1,\ 2,\ 3)$$

then (1.4) for ω = const gives

$$(A_{i+1} - A_{i+2})\,\omega^2 l_{i+1} l_{i+2} + l_{i+2} L_{i+1} - l_{i+1} L_{i+2} = 0 \qquad (1.8)$$

which, after multiplying L_i and rearranging , becomes

$$L_1 (A_2 - A_3)\, l_2 l_3 + L_2 (A_3 - A_1)\, l_3 l_1 + L_3 (A_1 - A_2)\, l_1 l_2 = 0 \qquad (1.9)$$

Direction cosines should also, apart from (1.9), satisfy

$$l_1^2 + l_2^2 + l_3^2 = 1 \qquad (1.10)$$

Since (1.9) and (1.10) possess no unique solution, there exists a number of axes satisfying the above equations. In terms of present coordinates l_i, point $S(l_1, l_2, l_3)$ should lie on both, the surface (1.9) and the phere (1.10), hence its geometrical locus is the intersection of (1.9) and (1.10). If the field is homogeneous

$$U = -mg\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3)$$

or Newtonian

$$U = -\alpha\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) - \beta\,(A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2)$$

the surface (1.9) will be a Staude cone [2]

$$x_0 (A_2 - A_3)\, l_2 l_3 + y_0 (A_3 - A_1)\, l_3 l_1 + z_0 (A_1 - A_2)\, l_1 l_2 = 0$$

where x_0, y_0, z_0 are the coordinates of the center of gravity of the body.

The sphere (1.10) and the surface (1.9) have common points $S_1 (\pm 1, 0, 0)$, $S_2 (0, \pm 1, 0)$, $S_3 (0, 0, \pm 1)$. These correspond to principal axes of inertia, provided the values of L_j are finite. Subscript j assumes the values 1, 2, 3, provided this value is different from that of i corresponding to the axis under consideration. (1.9) is also satisfied by the point $S_4(l_1^1, l_2^1, l_3^1)$ of the sphere for which the normal $n\,(L_1', L_2', L_3')$ to the equipotential surface and the radius-vector OS_4 are colinear, i.e. for which

$L_1'/l_1 = L_2'/l_2 = L_3'/l_3$. It can easily be verified that

$$S_5\left(\frac{L_1}{A_1N}, \frac{L_2}{A_2N}, \frac{L_3}{A_3N}\right), \qquad N = \pm\left(\frac{L_1^2}{A_1^2}+\frac{L_2^2}{A_2^2}+\frac{L_3^2}{A_3^2}\right)^{1/2} \tag{1.11}$$

also satisfies (1.9). Moreover, any point S_0 (l_1^0, l_2^0, l_3^0), on the sphere which is a neutral point of the field, i.e. $L_i^0 = 0$, satisfies (1.9).

For every point S on the spherical curve obtained by the intersection of (1.9) and (1.10), there exists a corresponding semiaxis OS , which may become an axis of steady rotation. For this, its direction cosines must be such, that (1.8) gives a positive value for ω^2, i.e.

$$\omega^2 = \frac{1}{A_2 - A_3}\left(\frac{L_3}{l_3} - \frac{L_2}{l_2}\right) = \frac{1}{A_3 - A_1}\left(\frac{L_1}{l_1} - \frac{L_3}{l_3}\right) = \frac{1}{A_1 - A_2}\left(\frac{L_2}{l_2} - \frac{L_1}{l_1}\right) \geqslant 0 \tag{1.12}$$

For the points S_1 , S_2, S_3 we have $\omega^2 = \infty$, hence, the principal axes of inertia cannot, in general, be the permanent axes. For S_4 and S_0 we have $\omega = 0$, and the body is in equilibrium.

In case of any of the points S_1 , S_2 or S_3 coinciding with S_0 or S_4 , e.g. $S_1 = S_0$, the principal axis of inertia x_1 becomes, as seen from (1.8), an axis of steady rotation with arbitrary angular velocity. When any of the principal axes of inertia is, at the same time, the axis of steady rotation, then the force function is found to possess the following partial derivative: $\partial U / \partial\gamma_i = \gamma_i f_i(\gamma_1, \gamma_2, \gamma_3)$ $(i = 1, 2, 3)$.

For S_5 , (1.12) gives

$$\omega^2 = \mp\left(\frac{L_1^2}{A_1^2}+\frac{L_2^2}{A_2^2}+\frac{L_3^2}{A_3^2}\right)^{1/2}$$

hence steady rotation is possible only for the semiaxes passing through S_5 and corresponding to the minus sign in (1.11).

2. Particular cases. (a) Let the ellipsoid of inertia be an ellipsoid of revolution and let us suppose that $A_1 = A_2 \neq A_3$. Then the relations (1.8) and (1.9) will become

$$(A_1 - A_3)\,\omega^2 l_2 l_3 + l_3 L_2 - l_2 L_3 = 0, \quad -(A_1 - A_3)\,\omega^2 l_3 l_1 + l_1 L_3 - l_3 L_1 = 0$$

$$l_2 L_1 - l_1 L_2 = 0 \tag{2.1}$$

$$l_3\,(l_2 L_1 - l_1 L_2) = 0 \tag{2.2}$$

From (2.2) we see that the surface (1.9) consists of a plane

$$l_3 = 0 \tag{2.3}$$

and a surface

$$l_2 L_1 - l_1 L_2 = 0 \tag{2.4}$$

a_1) The case $l_3 = 0$. From (2.1) we see that every axis $(l_1, l_2, 0)$ on the plane (2.3) satisfying (2.4), will be the axis of steady rotation with arbitrary angular velocity, if $L_3 = 0$. The condition (2.4) for the principal axis x_1 will be satisfied if $L_2 = 0$, and for x_2 , if $L_1 = 0$.

a_2) The case $l_3 \neq 0$. Here, the possible axes of steady rotation and the corresponding angular velocity, satisfy the condition

$$\omega^2 = \frac{1}{A_1 - A_3}\left(\frac{L_3}{l_3} - \frac{L_1}{l_1}\right) = \frac{1}{A_1 - A_3}\left(\frac{L_3}{l_3} - \frac{L_2}{l_2}\right) \geqslant 0 \tag{2.5}$$

From (2.1) we see that if $L_1 = L_2 = 0$, then the principal axis of inertia x_3 is also the axis of steady rotation with arbitrary angular velocity.

b) If the inertia ellipsoid is a sphere, i.e. $A_1 = A_2 = A_3$, then from (1.8) it follows that the only axes of steady rotation will be those, corresponding to the normals to the equipotential surface or to its neutral points.

c) Ellipsoid of inertia is not the ellipsoid of revolution, but the force function is of the type $U = U(\gamma_1, \gamma_2)$. Assuming $L_3 = 0$, Equations (1.8) and (1.9) become

$$l_3 [(A_2 - A_3) \omega^2 l_2 + L_2] = 0, \qquad l_3 [(A_3 - A_1) \omega^2 l_1 - L_1] = 0 \tag{2.6}$$

$$(A_1 - A_2) \omega^2 l_1 l_2 + l_2 L_1 - l_1 L_2 = 0, \qquad l_3 [L_1 (A_2 - A_3) l_2 + L_2 (A_3 - A_1) l_1] = 0 \tag{2.7}$$

In this case the surface (1.9) separates into the plane (2.3) and the surface

$$L_1 (A_2 - A_3) l_2 + L_2 (A_3 - A_1) l_1 = 0 \tag{2.8}$$

c_1) The case $l_3 = 0$. From (2.6) it follows that the possible axes of steady rotation and the corresponding angular velocity are subject to the condition

$$\omega^2 = \frac{1}{A_1 - A_2}\left(\frac{L_2}{l_2} - \frac{L_1}{l_1}\right) \geqslant 0 \tag{2.9}$$

The principal axis of inertia $x_1 (x_2)$ will also be the axis of steady rotation with arbitrary angular velocity if $L_2 = 0$ $(L_1 = 0)$.

c_2) The case $l_3 \neq 0$. The axis of steady rotation should satisfy (2.8) Possible axes of steady rotation and the corresponding angular velocity are found from

$$\omega^2 = \frac{1}{A_3 - A_1}\frac{L_1}{l_1} = - \frac{1}{A_2 - A_3}\frac{L_2}{l_2} = \frac{1}{A_1 - A_2}\left(\frac{L_2}{l_2} - \frac{L_1}{l_1}\right) \geqslant 0 \tag{2.10}$$

Principal axis of inertia x_3 is an axis of steady rotation with arbitrary angular velocity if $L_1 = L_2 = 0$, i.e. if it corresponds to the neutral point of the function $U(\gamma_1, \gamma_2)$.

d) Let $U = U(\gamma_1)$. Since in this case $L_2 = L_3 = 0$, (1.8) gives

$$(A_2 - A_3) \omega^2 l_2 l_3 = 0, \qquad l_3 [(A_3 - A_1) \omega^2 l_1 - L_1] = 0$$

$$l_2 [(A_1 - A_2) \omega^2 l_1 + L_1] = 0 \tag{2.11}$$

d_1) The case $l_2 \neq l_3 = 0$., Axis x_1 is an axis of steady rotation with arbitrary angular velocity.

d_2) The case $l_2 = 0$, $l_3 \neq 0$. Possible axes of steady rotation and the corresponding angular velocity satisfy

$$\omega^2 = \frac{L_1}{l_1 (A_3 - A_1)} \geqslant 0$$

x_3 will be an axis of steady rotation with arbitrary angular velocity if $L_1 = 0$, i.e. when it corresponds to the neutral point of the force function.

d_3) The case $l_2 \neq 0$, $l_3 \neq 0$. If $A_2 \neq A_3$, then the equilibrium is possible only when $L_1 = 0$. If $A_2 = A_3$, then the possible axes of steady rotation and corresponding angular velocity can be determined from

$$\omega^2 = \frac{L_1}{l_1 (A_2 - A_1)} \geqslant 0$$

If $l_1 = 0$ is a neutral point of the force function $(L_1 = 0)$ and $A_2 = A_3$, then steady rotations about any of the axes $(0, l_2, l_3)$, occur with an arbitrary angular velocity.

3. Stability of steady rotations. Let

$$p_i = \omega l_i, \qquad \gamma_i = l_i = \text{const}, \qquad (i = 1, 2, 3), \qquad \omega = \text{const} \tag{3.1}$$

be the boundary conditions for the particular solution of equations of motion (1.2), corresponding to the steady rotation of a rigid body. We shall assume that the motion (3.1) is smooth and shall investigate its stability. Substituting

$$p_i = \omega l_i + \xi_i, \qquad \gamma_i = l_i + \eta_i, \quad (i = 1, 2, 3) \tag{3.2}$$

in (1.2) gives us equations of perturbed motion, whose first integrals are

$$V_1 = A_1\xi_1^2 + A_2\xi_2^2 + A_3\xi_3^2 + 2\omega\,(A_1l_1\xi_1 + A_2l_2\xi_2 + A_3l_3\xi_3) - \\ - 2U\,(l_1+\eta_1,\ l_2+\eta_2,\ l_3+\eta_3) = \text{const} \tag{3.3}$$

$$V_2 = A_1\xi_1\eta_1 + A_2\xi_2\eta_2 + A_3\xi_3\eta_3 + \omega\,(A_1l_1\eta_1 + A_2l_2\eta_2 + A_3l_3\eta_3) + \\ + A_1l_1\xi_1 + A_2l_2\xi_2 + A_3l_3\xi_3 = \text{const}$$

$$V_3 = \eta_1^2 + \eta_2^2 + \eta_3^2 + 2\,(l_1\eta_1 + l_2\eta_2 + l_3\eta_3) = 0$$

Expanding $U\,(l_1+\eta_1, l_2+\eta_2, l_3+\eta_3)$ in powers of η_i, , we obtain

$$V_1 = \sum_{i=1}^{3} A_i\xi_i^2 + 2\omega\sum_{i=1}^{3} A_i l_i\xi_i - 2\sum_{i=1}^{3} L_i\eta_i - \sum_{\substack{i=1\\j=1}}^{3} L_{ij}\eta_i\eta_j + \varphi\,(\eta_1, \eta_2, \eta_3)$$

where

$$L_{ij} = \left(\frac{\partial^2 U}{\partial\gamma_i\,\partial\gamma_j}\right)_{\gamma_1=l_1,\ \gamma_2=l_2,\ \gamma_3=l_3}$$

and $\varphi(\eta_1, \eta_2, \eta_3)$ is a function containing these terms of the expansion, in which the power of η_i is geater than two.

The method of Chetaev [6] is used here to construct the Liapunov function, which is

$$V = V_1 - 2\omega V_2 + \lambda V_3 + {}^1/_4\,\mu\,V_3^2 = A_1\xi_1^2 + A_2\xi_2^2 + A_3\xi_3^2 + (\lambda + \mu l_1^2 - L_{11})\,\eta_1^2 + \\ + (\lambda + \mu l_2^2 - L_{22})\,\eta_2^2 + (\lambda + \mu l_3^2 - L_{33})\,\eta_3^2 - 2\omega\,(A_1\xi_1\eta_1 + A_2\xi_2\eta_2 + A_3\xi_3\eta_3) + \\ + 2\,[(\mu l_1l_2 - L_{12})\,\eta_1\eta_2 + (\mu l_1l_3 - L_{13})\,\eta_1\eta_3 + (\mu l_2l_3 - L_{23})\,\eta_2\eta_3] + \\ + f\,(\eta_1, \eta_2, \eta_3) = \text{const} \tag{3.4}$$

λ here is based on (1.8), and is

$$\lambda = A_1\omega^2 + \frac{L_1}{l_1} = A_2\omega^2 + \frac{L_2}{l_2} = A_3\omega^2 + \frac{L_3}{l_3} \tag{3.5}$$

where μ is an arbitrary constant, and $f(\eta_1, \eta_2, \eta_3)$ contains η_i only in powers greater than two.

The function V will be a positive-definite function of variables ξ_i, η_i if the quadratic part of V, i.e. the quadratic form $W = V - f\,(\eta_1, \eta_2, \eta_3)$. is positive-definite. By the Sylvester criterion, W and consequently V will be positive-definite if and only if

$$m_1 - A_1\omega^2 > 0, \qquad \begin{vmatrix} m_1 - A_1\omega^2 & k_{12} \\ k_{12} & m_2 - A_2\omega^2 \end{vmatrix} > 0 \tag{3.6}$$

$$\begin{vmatrix} m_1 - A_1\omega^2 & k_{12} & k_{13} \\ k_{12} & m_2 - A_2\omega^2 & k_{23} \\ k_{13} & k_{23} & m_3 - A_3\omega^2 \end{vmatrix} > 0 \tag{3.7}$$

where

$$m_i = \lambda + \mu l_i^2 - L_{ii}, \qquad k_{i,\,i+1} = \mu l_i l_{i+1} - L_{i,\,i+1} \tag{3.8}$$

With the conditions (3.6) and (3.7) satisfied, function V will be a sign-definite integral of the equations of perturbed motion and, by Liapunov's theorem on stability, nonperturbed motion (3.1) corresponding to steady rotations of a rigid body, will also be stable with respect to the variables p_i, γ_i .

If the axis of steady rotation passes through the neutral point $S_0\,(l_1^\circ, l_2^\circ, l_3^\circ)$ of the force function, i.e. $L_i = 0$, then as it was shown before, $\omega = 0$, and the body is in equilibrium. When $\mu = 0$, (3.7) gives us sufficient conditions of stability, which are

$$-L_{11} > 0, \qquad \begin{vmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{vmatrix} > 0, \qquad - \begin{vmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{vmatrix} > 0 \tag{3.9}$$

The above inequalities represent sufficient conditions for the force function to have a maximum at S_0, i.e. they coincide with the sufficient conditions of stability according to the Lagrange (Legendre-Dirichlet) criterion.

If the force function U is such that $L_{ij} = 0$ $(i \neq j)$, then the sufficient conditions for stability of the steady rotations are obtained from (3.6), and are

$$\mu l_1^2 + \frac{L_1}{l_1} - L_{11} > 0 \tag{3.10}$$

$$\mu\left[\left(\frac{L_2}{l_2} - L_{22}\right) l_1^2 + \left(\frac{L_1}{l_1} - L_{11}\right) l_2^2\right] + \left(\frac{L_1}{l_1} - L_{11}\right)\left(\frac{L_2}{l_2} - L_{22}\right) > 0$$

$$\mu\left[\left(\frac{L_2}{l_2} - L_{22}\right)\left(\frac{L_3}{l_3} - L_{33}\right) l_1^2 + \left(\frac{L_1}{l_1} - L_{11}\right)\left(\frac{L_3}{l_3} - L_{33}\right) l_2^2 + \right.$$
$$\left. + \left(\frac{L_1}{l_1} - L_{11}\right) \times \left(\frac{L_2}{l_2} - L_{22}\right) l_3^2\right] + \left(\frac{L_1}{l_1} - L_{11}\right)\left(\frac{L_2}{l_2} - L_{22}\right)\left(\frac{L_3}{l_3} - L_{33}\right) > 0;$$

If, for example, we have a homogeneous field of force

$$U = -P\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3),$$

where P is the weight of the body and $x_0\, y_0\, z_0$ are the coordinates of its center of gravity, then (3.10) gives us well known conditions due to Rumiantsev, on the stability of steady rotation of a heavy rigid body with one fixed point [3].

Apykhtin [5] investigated the problem of steady rotation of a rigid body with a fixed point for the case of Goriachev, namely $A_1 = A_2 = 2A_3$ with the force function

$$U = A_1\,[a\,(n-1)^{-1}\gamma_3^{1-n} + {}^1/_2\, b\,(\gamma_2^2 - \gamma_1^2) - c_1\gamma_1 - c_2\gamma_2]$$

where a, b, c_1, c_2 are constants and n is a positive integral number. For the case

$$\begin{aligned} L_1 &= -A_1\,(c_1 + bl_1), & L_{11} &= -A_1 b & & \\ L_2 &= -A_1\,(c_2 - bl_2), & L_{22} &= A_1 b & L_{ij} &= 0 \quad (i \neq j) \\ L_3 &= -A_1 a l_3^{-n}, & L_{33} &= A_1 n a l_3^{-n-1} & & \end{aligned}$$

sufficient conditions (3.10), give, e.g. for $\mu = 0$, the inequalities

$$-\frac{c_1}{l_1} > 0, \qquad -\frac{c_2}{l_2} > 0, \qquad -\frac{a}{l_3^{n+1}} > 0$$

4. Stability of steady rotations about the principal axes of inertia. We shall assume that the force function satisfies the condition $L_{ij} = 0$ $(i \neq j)$. If the force function has a neutral point corresponding to any of the principal axes of inertia, then this axis will be the axis of steady rotation with arbitrary angular velocity. Let for example $S_1(1, 0, 0)$ be a neutral point. In order to investigate the stability of steady rotation about the axis x_1, we shall construct a Liapunov function in the form of (3.4), and substitute into it $l_1 = 1$, $l_2 = l_3 = 0$ and $\lambda = A_1\omega^2$. (3.6) are the conditions of sign-definiteness of this function, hence they are also the sufficient conditions of its stability, and can easily be reduced to

$$\mu - L_{11} > 0, \qquad (A_1 - A_2)\,\omega^2 - L_{22} > 0, \qquad (A_1 - A_3)\,\omega^2 - L_{33} > 0 \tag{4.1}$$

a) Let $S_1(1, 0, 0)$ correspond to the maximum of the force function, i.e. $L_{ii} < 0$ $(i = 1, 2, 3)$. If $\mu = 0$, then from (4.1) we have:

a_1) The case $A_1 > A_2 \geqslant A_3$, when the motion is stable for any angular velocity, i.e. steady rotations about the principal axis of inertia corresponding to the maximum moment of inertia, are stable for any angular velocity.

a_2) The case $A_2 > A_1 > A_3$. From (4.1) we see that the sufficient condition of stability is the inequality

$$\omega^2 < \frac{-L_{22}}{A_2 - A_1} \tag{4.2}$$

a_3) The case $A_1 < A_2 \leqslant A_3$. From (4.1) it follows that the sufficient conditions of stability are the inequalities

$$\omega^2 < \frac{-L_{22}}{A_2 - A_1}, \qquad \omega^2 < \frac{-L_{33}}{A_3 - A_1} \tag{4.3}$$

b) Let $S_1(1, 0, 0)$ correspond to the minimum of the force function, i.e. $L_{11} > 0$. Choosing $\mu > L_{11}$ we see that the conditions (4.1) are satisfied only when x_1 is the axis of the largest moment of inertia $A_1 > A_2 \geqslant A_3$ and when at the same time,

$$\omega^2 > \frac{L_{22}}{A_1 - A_2}, \qquad \omega^2 > \frac{L_{33}}{A_1 - A_3} \tag{4.4}$$

c) Let the neutral point $S_1(1, 0, 0)$ correspond to neither the maximum nor the minimum of the force function. If $L_{11} \geqslant 0$, $L_{22} < 0$, $L_{33} < 0$, then the results are equal to those of (a), while if $L_{11} \leqslant 0$, $L_{22} > 0$ and $L_{33} > 0$ the results are equal to those of (b). If $L_{22} > 0$, $L_{33} < 0$, and L_{11} is arbitrary, then for $A_1 > A_2 \geqslant A_3$, sufficient condition of stability easily deduced from (4.1), is the inequality

$$\omega^2 > \frac{L_{22}}{A_1 - A_2} \tag{4.5}$$

while for $A_3 > A_1 > A_2$, the inequalities

$$\frac{L_{22}}{A_1 - A_2} < \omega^2 < \frac{-L_{33}}{A_3 - A_1} \tag{4.6}$$

When $A_2 > A_1 > A_3$, then the second of the (4.1) inequalities cannot be satisfied.

The axis $x_1(1, 0, 0)$ will also be the axis of steady rotation with arbitrary angular velocity in case, when x_1 is normal to the equipotential surface, i.e. when $L_1 \neq 0$, $L_2 = L_3 = 0$. To investigate the stability of steady rotation in this case, we shall construct a Liapunov function in form of (3.4), in which $l_1 = 1$, $l_2 = l_3 = 0$, $\lambda = A_1\omega^2 + L_1$. This function will be positive-definite in accordance with (3.6), if

$$\mu + L_1 - L_{11} > 0, \qquad (A_1 - A_2)\,\omega^2 + L_1 - L_{22} > 0$$
$$(A_1 - A_3)\,\omega^2 + L_1 - L_{33} > 0 \tag{4.7}$$

which are the sufficient conditions of stability of steady rotations in this case, are fulfilled.

As an example of the use of (4.7), we shall investigate the conditions of stability of steady rotations when the force function is of the type

$$U = -\alpha\,(A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2) \tag{4.8}$$

where $\alpha > 0$ is a constant. Apykhtin proved that if $\omega^2 \neq 2\alpha$ and the inertia ellipsoid is not spherical, then the principal axes of inertia are the only permanent axes. When the force function is of the type (4.8), then sufficient conditions of stability of steady rotation about the axis $x_1(1, 0, 0)$ are, by (4.7), the inequalities

$$\mu > 0, \quad (A_1 - A_2)(\omega^2 - 2\alpha) > 0, \quad (A_1 - A_3)\,(\omega^2 - 2\alpha) > 0 \tag{4.9}$$

from which it can be seen that if the axis x_1 is the axis of the maximum moment of inertia when $A_1 > A_2 \geqslant A_3$, then the inequality

$$\omega^2 > 2\alpha \tag{4.10}$$

is sufficient a condition of stability, while if x_1 is the axis of the minimum moment of inertia, then the sufficient condition is

$$\omega^2 < 2\alpha \tag{4.11}$$

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